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SOME REMARKS ON NON-SMOOTH TRANSFORMATIONS OF SPACE AND TIME FOR VIBRATING SYSTEMS WITH RIGID BARRIERS†

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Two different realizations of the idea of a non-smooth transformation of variables for systems with rigid barriers (constraints) are discussed. It is shown that non-smooth space transformations and non-smooth time transformations are mutually complementary. They may also be applied to the same system simultaneously, forming a composition. © 2002 Elsevier Science Ltd. All rights reserved.

The method of a non-smooth coordinate transformation (henceforth: ST = space transformation) for systems with absolutely rigid unilateral constraints was proposed in [1, 2] and documented in [3]. On the other hand, it was later shown that, for a special class of oscillating systems without any constraints, one can introduce a non-smooth time transformation (TT) in the form of a sawtooth sine curve ([4], "oscillating time"). Since a sawtooth sine curve is not invertible over a whole period, direct introduction of sawtooth time, like any other periodic time, requires the corresponding time symmetry of a periodic process. This restriction was eliminated in [5] by a special complexification of the coordinates. It should be mentioned that a TT inherits the abstract constituent of the idea of a transformation of variables by non-smooth functions, and in many cases operates with the same piecewise-linear functions as an ST. In addition, the prehistory of the construction of TTs stemmed from the attempt to adapt STs for the special case of "weakened constraints," namely, an extremely non-linear oscillator with a power characteristic, which is equivalent to rigid unilateral constraints in the limiting case of an infinite exponent [6]. Nevertheless, it will be shown below that STs and TTs differ both in the physical sense and in their mathematical formulation, as well as in the classes of systems and dynamical regimes for which these methods are primarily designed. In more detailed terms:

1. STs expand the space and do not affect time, while TTs contract time, generating special algebraic structures (hyperbolic numbers) in the space.
2. From the standpoint of mechanics, the purpose of using STs is to obtain rigid barriers, which, however, creates unsatisfactory situations for the iterative algorithms that often follow a TT.
3. STs do not depend, for the most part, on dynamical regimes, whereas TTs require the process to be periodic or, at least, capable of oscillating.
4. Equations transformed by STs and TTs differ substantially in form, since an ST is a strongly non-linear transformation of the space, while a TT generates a linear coordinate transformation.
5. From a function-theoretical standpoint, STs are applied to functions (images), while TTs transform arguments (pre-images).

For all that, the TT method was applied in [7, 8] to a system with rigid constraints after preliminary processing. Possibly, owing to some inconsistency with the second of the above remarks, the TT was accompanied by a reference to [1, 2], while the paper [5] was qualified as having to do with "vibroimpact systems." (In actual fact, "vibroimpact systems" were not considered at all in [5], and no time transformation was introduced in [1, 2].) The explanation is probably that in [7, 8] only periodic modes were considered, in which the action of the barriers may be described by singular periodic functions which depend explicitly on time. (Note that such a modelling device has also been considered before [9] from a more general standpoint, irrespective of any transformation of variables.) After such a preliminary step, the use of a TT makes it possible to eliminate singular functions of time and to formulate a boundary-value problem with respect to the new time variable [10].

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The main result of this paper is formulated below in Section 3, where we propose a composition of two transformations, TT∘ST, which essentially illustrates their mutually complementary nature. In Sections 1 and 2 we present a preliminary illustration of each transformation separately, as applied to the simplest physical objects.

1. THE SPACE TRANSFORMATION

Let us consider a linear oscillator between two absolutely rigid barriers, driven by an external harmonic load. The Lagrangian, reduced to unit mass, and the equation of the constraints may be represented as follows (the dot denotes differentiation with respect to time)

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega_0^2}{2}x^2, \quad -1 \leqslant x \leqslant 1 \tag{1.1}$$

where the meaning of the parameters and the variables is explained by the form of the Lagrangian; the barriers are assumed to be ideally elastic, so that the coefficient of restitution is unity.

According to the idea of the ST, the constraints are eliminated by a change of coordinate

$$x(t) \rightarrow l(t): \quad x = S(l) \tag{1.2}$$

where S is a sawtooth piecewise-linear sine function of unit amplitude and period equal to four. This implies the truth of the relation $[dS(l)dl]^2 = 1$ at least for almost all l, and the inequality in (1.1) is automatically satisfied. Substituting expression (1.2) into system (1.1), we obtain a transformed system in which there are now no constraints, but the configuration space becomes infinite

$$L = \frac{1}{2}\dot{l}^2 - \frac{\omega_0^2}{2}S^2(l), \quad -\infty < l < \infty \tag{1.3}$$

Note that, for convenience in further comparisons, the normalization of the period and the notation have been altered compared with the original papers [1, 2]. In addition, the initial version of the transformation was formulated in terms of Newtonian mechanics, although Zhuravlev also described the transformation of the Lagrangian in one of the discussions. The latter formulation enables one to avoid introducing the so-called "impact terms" into the equations of motion. In fact, the basic role of STs is precisely to give a correct formulation of the differential equation of motion for a system with constraints, i.e. of an equation which is true over the entire time interval of the motion.

The corresponding differential equation of motion has the form

$$\ddot{l} + \omega_0^2(dS(l)/dl)S(l) = 0 \tag{1.4}$$

This is an extremely non-linear system, whose characteristic has a periodic sequence of discontinuities of the first kind, generated by the singularities of the derivative $dS(l)/dl$. Such singularities are dealt with, however, within the confines of the theory of generalized functions, which treats equalities as integral identities. Indeed, Eq. (1.4) follows from the variational principle

$$\delta\int_{t_1}^{t_2} L[l(t),\ \dot{l}(t)]dt = -\int_{t_1}^{t_2}\left[\ddot{l}(t) + \omega_0^2\frac{dS(l)}{dl}S(l)\right]\delta l dt = 0$$

that is, in the intermediate stage of the derivation, the equation of motion has the form of an integral identity. The left-hand side of the equation of motion (1.4) is then considered as a force which performs zero work on the displacements δl. The essence of this generalized approach is that the force itself is not necessarily equal to zero at each individual point t. Thus, the generalized treatment of the equations of motion enables us to ignore those times at which the derivative $dS(l(t))dl(t)$ has no definite meaning.

2. THE TIME TRANSFORMATION

On the other hand, let us consider the same example (1.1), but for the moment dropping the restriction on the coordinate $x(t)$ and beginning with the equation in its Newtonian form

$$\ddot{x} + \omega_0^2 x = 0 \tag{2.1}$$

Note that there is no non-smoothness, both in the equation and in its obvious solution. Nevertheless, a non-smooth (sawtooth) transformation of time is perfectly applicable and may be written as

$$t \to \tau: \quad \tau = S(\omega t) \tag{2.2}$$

$$x = X(\tau) + Y(\tau)e, \quad e^2 = 1; \quad e = dS(\omega t)/d(\omega t) \tag{2.3}$$

where the scaling factor ω of the leading time variable remains to be determined, and we have also introduced notation for a right-angled cone, expressed in terms of the generalized derivative of a sawtooth sine. If the function $x(t)$ is given, then the functions $X(\tau)$ and $Y(\tau)$ are known [5, Remark 1]; otherwise, they have to be found from the transformed problem.

In many cases, to which Eq. (2.1) also relates, one has a single-component representation of the solution, such as the following

$$x = X(\tau) \tag{2.4}$$

Substitution of this expression into Eq. (2.1) gives the following relation (the prime denotes differentiation with respect to τ)

$$\omega^2 \left[X'' + X' \frac{de(\omega t)}{d(\omega t)} \right] + \omega_0^2 X = 0 \tag{2.5}$$

and, in the final analysis, we obtain a boundary-value problem in terms of the new time variable (compare with the outcome of the ST (1.4))

$$\omega^2 X'' + \omega_0^2 X = 0 \tag{2.6}$$

$$X'(\pm 1) = 0 \tag{2.7}$$

where the boundary conditions serve to exclude the periodic singular term in Eq. (2.5), which contains the derivative

$$\frac{de(\omega t)}{d(\omega t)} = \frac{d^2 S(\omega t)}{d(\omega t)^2} = 2 \sum_{k=-\infty}^{\infty} [\delta(\omega t + 1 - 4k) - \delta(\omega t - 1 - 4k)]$$

The eigenvalues of problem (2.6), (2.7) are defined by the relation

$$\omega = \frac{2\omega_0}{j\pi}, \quad j = 1, 2, \ldots \tag{2.8}$$

and the single-parameter family of solutions for the harmonic oscillator (2.1) takes the form

$$x = X(\tau) = A \sin\left(\frac{j\pi\tau}{2} + \varphi_j \right), \quad \varphi_j = \frac{\pi}{4}[1 + (-1)^j] \tag{2.9}$$

where A is an arbitrary amplitude and $t = S(\omega(t))$ is a sawtooth "oscillating time".

The effect of quantizing the parameter ω in this case has no significance, since for any j we have the following identity in t

$$\sin\left[\frac{j\pi}{2} S\left(\frac{2\omega_0 t}{j\pi} \right) + \varphi_j \right] = \sin(\omega_0 t + \varphi_j), \quad -\infty < t < \infty \tag{2.10}$$

(this quantization, however, may be given a physical meaning when one is considering combination resonances in the case of non-linear forced oscillations).

The general solution may be obtained either by introducing a constant phase shift, corresponding to the invariance of the equation under a translation group with respect to the initial time variable, or by using a general form of representation for the desired solution (2.3). In the latter case, the general solution has the form

$$x = A\sin\left(\frac{j\pi\tau}{2} + \varphi_j\right) + B\cos\left(\frac{j\pi\tau}{2} - \varphi_j\right)e \tag{2.11}$$

where A and B are arbitrary constants. This last expression explains the types of time symmetry of the X- and Y-components in the general representation for the periodic function (2.3).

We wish to ascertain whether a TT can be applied directly to a constrained system (1.1). Formally, this simply means adding the restriction $-1 \leq X(t) \leq 1$ to boundary-value problem (2.6), (2.7), since in this case the space coordinate has no part at all in the transformation. Thus, a TT is applicable, but (unlike the case of an ST) the constraints still cannot be removed.

3. COMPOSITION OF THE SPACE AND TIME TRANSFORMATIONS

We will now show that a system transformed according to an SR admits of a further transformation by the TT method. All the essential features of the composition may be amply illustrated in the case of a non-linear oscillator placed between two ideally elastic and absolutely rigid barriers and driven by a periodic external force. The differential equation for the motions between the barriers may be written in the form

$$\ddot{x} + 2\zeta\dot{x} + f(x) = F\sin(\Omega t + \alpha) \tag{3.1}$$

and the condition of the constraints may be expressed, as before, in the form

$$-1 \leq x \leq 1 \tag{3.2}$$

At the first step, we make the substitution (1.2) in Eq. (3.1). According to the idea of an ST, the singular term $(d^2S(l)/dl^2)\dot{l}^2$ then appearing must simply be dropped together with condition (3.2), since this term alone in the acceleration may be caused by the reaction of the constraints (impacts). After this, the result of the transformation becomes

$$\ddot{l} + 2\zeta\dot{l} + f(S(l))dS(l)/dl = F\sin(\Omega t + \alpha)dS(l)/dl \tag{3.3}$$

where $l(t)$ is the new coordinate of the system ($-\infty < l < \infty$).

Proceeding to the second step, we note that the initial version of the TT (2.3) may be applied only to a set of periodic solutions. It is therefore preferable to consider a TT as a transformation of dynamical regimes, not of the system itself. Thus, let us assume that the motion is periodic and that it has basic frequency Ω, that is, the same as the external force.

We introduce a sawtooth time variable by the relations

$$\tau = S(\omega t), \quad l = X(\tau) + Y(\tau)e, \quad \omega = (2/\pi)\Omega \tag{3.4}$$

(the appearance of the coefficient $2/\pi$ is due to the different normalization of the period of the sawtooth and standard sines).

The function of the external load must also be expressed in terms of the new argument, as follows:

$$F\sin(\Omega t + \alpha) = F\cos\alpha\sin\frac{\pi\tau}{2} + F\sin\alpha\cos\frac{\pi\tau}{2}e \tag{3.5}$$

This last expression may be obtained either by suitable modification of solution (2.11), or by applying a remark made in [5] and relating in general to any periodic function. Next, substituting (3.4) and (3.5) into Eq. (3.3) and following the TT scheme, we obtain

$$\omega^2X'' + 2\zeta\omega Y' + R_f(X,Y) + [\omega^2Y'' + 2\zeta\omega X' + I_f(X,Y)]e + \\ \underline{+\omega^2X'\frac{de(\omega t)}{d(\omega t)}} = R_F(X,Y,\tau) + I_F(X,Y,\tau)e \tag{3.6}$$

where

$$\left.\begin{matrix} R_f(X,Y) \\ I_f(X,Y) \end{matrix}\right\} = \frac{1}{2}\left\{\frac{dS(X+Y)}{d(X+Y)} f[S(X+Y)] \pm \frac{dS(X-Y)}{d(X-Y)} f[S(X-Y)]\right\}$$

$$\left.\begin{matrix} R_F(X,Y,\tau) \\ I_F(X,Y,\tau) \end{matrix}\right\} = \frac{F}{2}\left\{\frac{dS(X+Y)}{d(X+Y)} \sin\left(\frac{\pi\tau}{2}+\alpha\right) \pm \frac{dS(X-Y)}{d(X-Y)} \sin\left(\frac{\pi\tau}{2}-\alpha\right)\right\}$$

and the necessary continuity condition for $l(t)$ has the form

$$Y(\pm 1) = 0 \tag{3.7}$$

Essential use was made in deriving Eq. (3.6) of the algebraic properties of the expressions which guarantee the validity of the relation $e^2 = 1$. The underlined singular term must be eliminated by using the necessary condition for $l(t)$ to be differentiable, which has the form of (2.7).

Note that if the external load had included a periodic sequence of impulses, expressed in terms of $de(\omega t)/d(\omega t)$, the underlined term in (3.6) would have made it possible to eliminate the sequence; in the latter case, the boundary conditions (2.7) would have been inhomogeneous.

Now, the remaining E- and I-parts of Eq. (3.6) yield a system

$$\begin{aligned} \omega^2 X'' + 2\zeta\omega Y' + R_f(X,Y) &= R_F(X,Y,\tau) \\ \omega^2 Y'' + 2\zeta\omega X' + I_f(X,Y) &= I_F(X,Y,\tau) \end{aligned} \tag{3.8}$$

which, together with the boundary conditions (3.7) and (2.7), represents the final result of the composition of the transformations TT∘ST. Figure 1 shows the corresponding physico-geometrical treatment of this superposition, on the left are the initial space-time coordinates, and on the right, the result of successive application of the two transformations.

If the solution of the boundary-value problem is obtained, then, in terms of the initial variables of Eq. (3.1), the solution is expressed as

$$x(t) = S[X(S(\omega t)) + Y(S(\omega t))dS(\omega t)/d(\omega t)] \tag{3.9}$$

In the special case of zero damping ($\zeta = 0$) and an oscillator with an odd characteristic ($f(-x) = -f(x)$), the boundary-value problem may be presented in the much simpler form

$$\omega^2 X'' + \frac{dS(X)}{dX} f[S(X)] = F\frac{dS(X)}{dX}\sin\frac{\pi\tau}{2}, \quad Y \equiv 0; \quad X'(1) = 0, \quad X(-\tau) = -X(\tau) \tag{3.10}$$

is $\alpha = 0$, or in the form

$$\omega^2 Y'' + \frac{dS(Y)}{dY} f[S(Y)] = F\frac{dS(Y)}{dY}\cos\frac{\pi\tau}{2}, \quad X \equiv 0; \quad Y(1) = 0, \quad Y(-\tau) = Y(\tau) \tag{3.11}$$

if $\alpha = \pi/2$.

Note that, by formula (3.9), the superposition of solutions corresponding to these special cases is incorrect, because of the non-linearity of the problem.

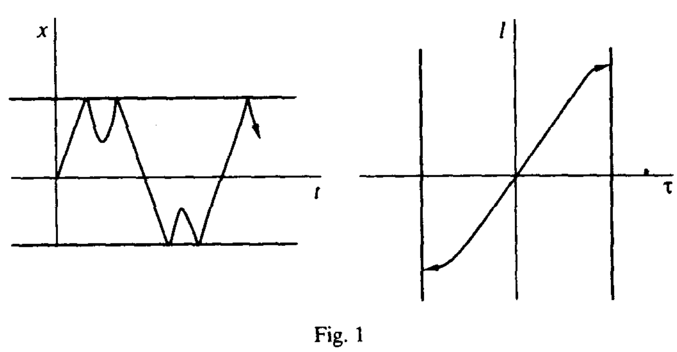

Fig. 1

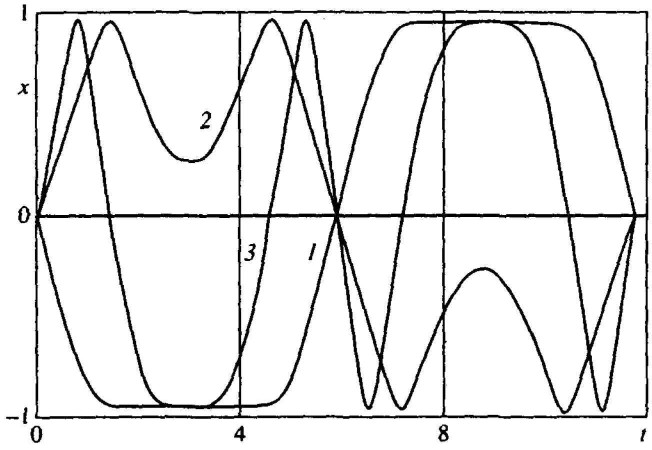

Fig. 2

The question of the existence and uniqueness of solutions for the non-linear boundary-value problem obtained above must be addressed to the appropriate branch of the mathematical theory. Examples of constructive iterative procedures for explicit analytical solutions may be found in the publications cited here. Another of the possible steps following the transformation is to express the problem in integral form and apply standard numerical codes. Finally, numerical solutions may be obtained directly by using the shooting method. When that is done one should note that the transformed problem is formulated in the interval $-1 \leqslant \tau \leqslant 1$, corresponding to half a period in terms of the original time variable. (In the general case, this is unfortunately achieved at the cost of doubling the number of equations.)

Depending on the choice of the computation scheme, it may be necessary to smooth out the singularities of the sawtooth sine. This may be done, for example, by considering the smooth family of functions

$$S_\eta(t) = \frac{2}{\pi}\arcsin\left(\eta \sin\frac{\pi t}{2}\right), \quad 0 < \eta \leqslant 1 \tag{3.12}$$

If $\eta = 1$, one has the sawtooth sine: $S_1(t) = S(t)$. But if the parameter η is slightly less than one, then the function $S\eta(t)$, remaining close to the sawtooth point for point, becomes smooth. From a physical standpoint, this device may be treated in a certain sense as weakening the rigid constraints. In other words, the barriers need no longer be considered to be absolutely rigid, which is indeed in better agreement with reality.

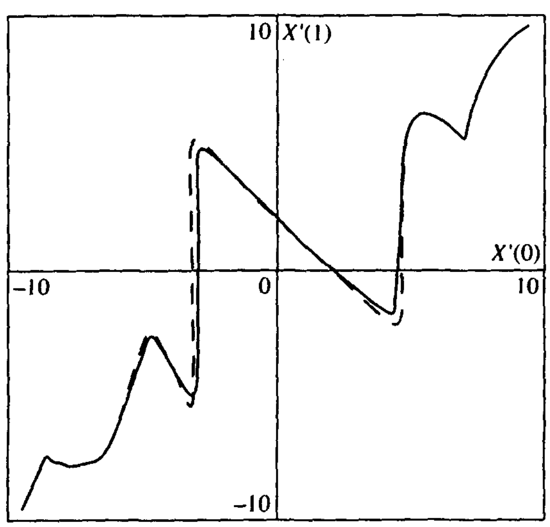

Fig. 3

As an example, boundary-value problem (3.10) was considered for the characteristic $f(x) = x$ and the following parameter values: $F = 1.2$, $\Omega = 0.6$, $\eta = 0.99$. In Fig. 2 we show three solutions (curves 1, 2 and 3), corresponding to three different values of the derivative $X'(0) = -3.09$, 2.31 and 4.75, which ensure that the boundary condition $X'(1) = 0$ is satisfied; this corresponds to the idea of the shooting method. These values are shown in Fig. 3 as the points of intersection of the curves with the horizontal coordinate axis. The corresponding values of the initial velocity of the oscillator may be computed as $x(0) = (2\Omega/\pi)X'(0)$. For comparison, these curves are shown for the values $\eta = 0.99$ (the continuous curve) and $\eta = 0.9999$ (the dashed curve).

In conclusion, we note that non-smooth transformations of the space and time variable are generally not commutative, that is, the transformations TT∘ST and ST∘TT yield expressions of different forms.

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